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Extremes of Shepp statistics for Gaussian random walk

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Abstract

Let $(\xi_i, i \geq 1)$ be a sequence of independent standard normal random variables and let $S_k = \sum_{i=1}^{k} \xi_i$ be the corresponding random walk. We study the renormalized Shepp statistic $M_T^{(N)} = \frac{1}{\sqrt{N}} \max_{1 \leq k \leq TN} \max_{1 \leq L \leq N} (S_{k+L-1} - S_{k-1})$ and determine asymptotic expressions for the probability $\mathbf{P}\left(M_T^{(N)} > u\right)$ when u, N and $T \to \infty$ in a synchronized way. There are three types of relations between u and N that give different asymptotic behavior. For these three cases we establish the limiting Gumbel distribution of $M_T^{(N)}$ when $T, N \to \infty$ and present corresponding normalization sequences.

Keywords: Gaussian random walk, increments, maximum, extreme values, high excursions, large deviations, moderate deviations, asymptotic behavior, Shepp statistics, distribution tail, Gumbel law, limit theorems, weak theorems, Wiener process.

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1 Introduction

Let $(\xi_i, i \geq 1)$ be a sequence of independent standard normal random variables, and let $S_k = \sum_{i=1}^{k} \xi_i$, with $S_0 = 0$, be the corresponding random walk. Introduce the Shepp and the closely related Erdös-Rényi statistics

$$W_{N,L} = \max_{1 \leq l \leq L} T_{N,l} \quad \text{and} \quad T_{N,L} = \max_{1 \leq k \leq N} S_{k+L-1} - S_{k-1},$$

and define

$$\zeta_L^{(N)}(k) = \frac{1}{\sqrt{N}} (S_{k+L-1} - S_{k-1}) = \frac{1}{\sqrt{N}} \sum_{i=k}^{k+L-1} \xi_i.$$

We study the asymptotic behavior of the probability

$$\mathbf{P}\left(\max_{\substack{0 < k \leq TN \\ 0 < L \leq N}} \zeta_L^{(N)}(k) > u \right) \tag{1}$$

when $u \to \infty$, $N \to \infty$ in a coordinated way. In fact, (1) is the probability of exceeding the level $u\sqrt{N}$ by the Shepp statistic $W_{TN,N}$. Related problems were described in Erdös and Rényi (1970),

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Piterbarg (1991), Kozlov (2004) and Piterbarg and Kozlov (2002). Paper Piterbarg (1991) presents the asymptotic behavior of the probability of moderate deviations for the Erdös-Rényi statistic under the assumption of sub-gaussian distribution of random walk step and papers Kozlov (2004) and Piterbarg and Kozlov (2002) study large deviations of the Erdös-Rényi and Shepp statistics for Cramér random walk. To get the full picture of all possible cases of asymptotic behavior of (1) we reformulate the result obtained by A.M. Kozlov in Kozlov (2004). Let $\psi(u) = \frac{1}{\sqrt{2\pi}} \int\limits_u^\infty e^{-x^2/2} dx$ be the tail of standard normal distribution and introduce the finite positive constant

$$J_\theta = \lim_{l \to \infty} \frac{1}{\theta l} \mathbf{E} exp \left\{ \theta \max_{0 \le n < l} (\sqrt{2} S_n - \theta n) \right\}.$$

Theorem 1.1 (A.M. Kozlov). *Assume* $u \to \infty,\ N \to \infty, \frac{u}{\sqrt{N}} \to \theta,$ *where* $0 < \theta < \infty$. *If* $Tu^2 \psi(u) \to 0$ *and* $Tu^2 \to \infty$, *then*

$$\mathbf{P} \left(\max_{\substack{0 < k \le TN \\ 0 < L \le N}} \zeta_L^{(N)}(k) > u \right) \sim J_\theta T u^2 \psi(u).$$

The present paper extends this result to moderate and excessively large deviations. For comparison and ease of reference we also restate the main result of Zholud (2008) which deals with the continuous time case and is crucial in proving the asymptotic formula for the case of moderate deviations. Let $W(\cdot)$ be the standard Brownian motion.

Theorem 1.2 (D. Zholud). *Assume* $u \to \infty$. *If* $Tu^2 \to \infty$ *and* $Tu^2 \psi(u) \to 0$, *then*

$$\mathbf{P} \left(\max_{\substack{0 \le t \le T \\ 0 \le s \le 1}} (W(t+s) - W(t)) > u \right) = HTu^2 \psi(u)(1 + o(1)),$$

where the constant

$$H = \lim_{B \to \infty} \lim_{A \to \infty} A^{-1} e^{-\frac{A+B}{2}} \mathbf{E} \exp \left(\max_{\substack{0 \le t \le A \\ 0 \le s \le B}} (W(t+s+A) - W(t)) \right)$$

is finite and positive.

The case of moderate deviations (i.e. $\frac{u}{\sqrt{N}} \to 0$ when $u \to \infty$) is intermediate between Theorem 1.1 and Theorem 1.2.

Theorem 1.3. *Assume* $u \to \infty,\ N \to \infty,\ \frac{u}{\sqrt{N}} \to 0$. *If* $Tu^2 \to \infty$ *and* $Tu^2 \psi(u) \to 0$, *then*

$$\mathbf{P} \left(\max_{\substack{0 < k \le TN \\ 0 < L \le N}} \zeta_L^{(N)}(k) > u \right) \sim HTu^2 \psi(u).$$

Indeed, this asymptotic behavior is different from the one in Theorem 1.1 in the constant multiplier and coincides with the asymptotic behavior for the case of continuous time, Theorem 1.2. The proof of Theorem 1.3 can be found in Section 2.

A further comment is that if $N \to \infty$ and u is fixed, then we could apply weak convergence of a random walk to the Wiener process, and the probabilities in Theorem 1.2 and Theorem 1.3 would coincide. However Section 3 shows that just applying weak convergence under the probability sign leads to incorrect results, and that the rigorous and somewhat technical proof of Theorem 1.3 is indeed needed. The main result of this section is as follows.

Theorem 1.4. *Assume $u \to \infty$, $N \to \infty$, $\frac{u}{\sqrt{N}} \to \infty$. If $TN \geq 1$ and $TN\psi(u) \to 0$, then*

$$\mathbf{P}\left(\max_{\substack{0<k\leq TN\\0<L\leq N}} \zeta_L^{(N)}(k) > u\right) \sim [TN]\psi(u).$$

This theorem completes full description of the possible asymptotic behavior of (1) under various relations between u and N.

Finally, using the results of Sections 2 and 3 we obtain limit Gumbel distribution for $M_T^{(N)}$ when $T, N \to \infty$. If one of the following relations holds,

$$1)\frac{2\ln T}{N} \to 0. \quad 2)\frac{2\ln T}{N} \to \theta^2 > 0. \quad 3)\frac{2\ln T}{N} \to \infty,$$

then, there exist functions a_T and b_T such that for any fixed x

$$\mathbf{P}\left(\max_{\substack{0<k\leq TN\\0<L\leq N}} a_T(\zeta_L^{(N)}(k) - b_T) \leq x\right) = e^{-e^{-x}} + o(1).$$

The corresponding theorems and normalizing constants can be found in Section 4. A similar result for standardized increments of Gaussian random walk is obtained in Kabluchko (2007).

There is also extensive literature on a.s. convergence of related quantities, see e.g. Shepp (1964), Erdös and Rényi (1970) and Frolov (2004).

2 Moderate deviations of the Shepp statistic

In this section we prove Theorem 1.3. That is we find the asymptotic behavior of the probability (1) when $u \to \infty$ and $u/\sqrt{N} \to 0$. It will be shown that it coincides with the asymptotic behavior for continuous time case, given in Theorem 1.2. The idea of the proof is similar to Zholud (2008) and we divide it into two main parts.

First, for any positive constant B we will focus on the asymptotic behavior of maximum of $\zeta_L^{(N)}(k)$ over a thin layer

$$\{(k, L) : 0 < k \leq TN, (1 - Bu^{-2})N < L \leq N\}.$$

Within this area and for large u, $\zeta_L^{(N)}(k)$ behaves approximately like $\zeta_N^{(N)}(k)$, and it will be shown that it is possible to determine the asymptotic behavior using similar techniques as used for stationary process in Piterbarg (1991).

Second, knowing the asymptotic behavior of maximum of the random variable $\zeta_L^{(N)}(k)$ over the area of its maximum variance, we will show that the maximum over the complementary set $\{(k, L) : 0 < k \leq TN, 0 < L \leq (1 - Bu^{-2})N\}$ gives a neglible contribution to the probability in (1).

The proof of the first part is based on the Double Sum Method. The lemma below is the analog of Lemma 2.1 in Zholud (2008). Let A and B be positive constants and set $p = Au^{-2}$, $q = Bu^{-2}$. By $A_0(u)$ we refer to the set of pairs $(k, L) \in [0, pN] \times ((1-q)N, N]$, where k and L are positive integers.

Lemma 2.1. *Let $u \to \infty$. Then*

$$\mathbf{P}\left(\max_{A_0(u)} \zeta_L^{(N)}(k) > u\right) = H_A^B \frac{1}{\sqrt{2\pi}u} e^{-\frac{u^2}{2}}(1 + o(1)), \tag{2}$$

where

$$H_A^B = e^{-\frac{A+B}{2}} \mathbf{E} \exp\left(\max_{\substack{0\leq t\leq A\\0\leq s\leq B}} W(t + s + A) - W(t)\right).$$

Proof: Let $[x]$ denote the integer part of x. We have

$$\max_{A_0(u)} \zeta_L^{(N)}(k) = \max_{A_0(u)} \frac{1}{\sqrt{N}} \sum_{i=k}^{[k+L-1]} \xi_i = \frac{1}{\sqrt{N}} \sum_{i=[pN]+1}^{[(1-q)N]} \xi_i + \max_{A_0(u)} \frac{1}{\sqrt{N}} \left(\sum_{i=k}^{[pN]} \xi_i + \sum_{i=[(1-q)N]+1}^{k+L-1} \xi_i \right).$$

The $L + [pN] - [(1-q)N]$ random variables in the sums inside the “max” sign are independent of the variables in the sum outside the “max” sign. We renumber the variables inside the maximum sign and denote them by ξ_i'. Thus,

$$\begin{aligned}\mathbf{P}\left(\max_{A_0(u)} \zeta_L^{(N)}(k) > u\right) &= \mathbf{P}\left(\frac{1}{\sqrt{N}} \sum_{i=[pN]+1}^{[(1-q)N]} \xi_i + \max_{\substack{0<k\leq pN \\ 0<L\leq qN}} \frac{1}{\sqrt{N}} \sum_{i=k}^{k+L+[pN]-1} \xi_i' > u\right) \\ &= \int_{-\infty}^{\infty} \frac{e^{-\frac{v^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} \mathbf{P}\left(\max_{\substack{0<k\leq pN \\ 0<L\leq qN}} \frac{1}{\sqrt{N}} (S'_{k+L+[pN]-1} - S'_{k-1}) > u - v\right) dv,\end{aligned}$$

where $\sigma^2 = \frac{[(1-q)N]-[pN]}{N}$ and $S_k' = \sum_{i=1}^{k} \xi_i'$ with $S_0' = 0$.

For the sake of briefness introduce

$$M(k,L) = \max_{\substack{0<k\leq pN \\ 0<L\leq qN}} \frac{1}{\sqrt{pN}} (S'_{k+L+[pN]-1} - S'_{k-1}).$$

Using the change of variables $v = u - \frac{\sqrt{A}w}{u}$, and recalling that $u\sqrt{p} = \sqrt{A}$, the probability in question is seen to equal to

$$\frac{\sqrt{A}}{\sqrt{2\pi\sigma^2}u} \int_{-\infty}^{\infty} e^{-\frac{(u-\sqrt{A}w/u)^2}{2\sigma^2}} \mathbf{P}(M(k,L) > w)\, ds = \frac{\sqrt{A}}{\sqrt{2\pi\sigma^2}u} e^{-\frac{u^2}{2\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{Aw^2/u^2}{2\sigma^2}} e^{\frac{\sqrt{A}w}{\sigma^2}} \mathbf{P}(M(k,L) > w)\, dw. \quad (3)$$

By weak convergence of a random walk to the Wiener process, for any w,

$$\lim_{pN\to\infty} \mathbf{P}(M(k,L) > w) = \mathbf{P}\left(\max_{\substack{0\leq t\leq 1 \\ 0\leq s\leq B/A}} W(t+s+1) - W(t) > w\right),$$

where $W(\cdot)$ is the standard Wiener process; using Lemma 1 Piterbarg (1991) it is straightforward to show that

$$\mathbf{P}(M(k,L) > w) \leq 2e^{-\frac{w^2}{24}}.$$

Thus, by dominated convergence

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-\frac{Aw^2/u^2}{2\sigma^2}} e^{\frac{\sqrt{A}w}{\sigma^2}} \mathbf{P}(M(k,L) > w)\, dw &= \int_{-\infty}^{\infty} e^{\sqrt{A}w} \mathbf{P}\left(\max_{\substack{0\leq t\leq 1 \\ 0\leq s\leq B/A}} (W(t+s+1) - W(t)) > w\right) dw + o(1) \\ &= \frac{1}{\sqrt{A}} \mathbf{E} \exp\left(\max_{\substack{0\leq t\leq A \\ 0\leq s\leq B}} W(t+s+A) - W(t)\right) + o(1).\end{aligned}$$

Finally, since $\sigma^2 = 1 - p - q + o(u^{-2})$ the factor in front of the integral (3) is equal to

$$\frac{\sqrt{A}}{\sqrt{2\pi}u} e^{-\frac{u^2}{2}(1+p+q+o(u^{-2}))}(1+o(1)) = \frac{1}{\sqrt{2\pi}u} e^{-\frac{u^2}{2}} \sqrt{A} e^{-\frac{A+B}{2}} (1+o(1))$$

and we obtain (2). □

Our next aim is to consider the layer $[0, TN] \times ((1-q)N, N]$. We use Lemma 2.1 and the Bonferroni inequality to obtain estimates of the probability of high level excursions of the maximum of $\zeta_L^{(N)}(k)$. Then we will show that estimates from below and from above are asymptotically equivalent.

Define $\Delta_k(u) = \{kpN+1, ..., (k+1)pN\} \times \{(1-q)N+1, ..., N\}$. For ease of notation we suppress dependence on u and assume that pN and qN are integers. Using stationarity of $\zeta_L^{(N)}(k)$ with respect to k, we obtain that

$$(Tp^{-1}+1)\mathbf{P}\left(\max_{\Delta_0} \zeta_L^{(N)}(k) > u\right) \geq \mathbf{P}\left(\max_{\substack{0<k\leq TN \\ (1-q)N<L\leq N}} \zeta_L^{(N)}(k) > u\right)$$

$$\geq (Tp^{-1}-1)\mathbf{P}\left(\max_{\Delta_0} \zeta_L^{(N)}(k) > u\right)$$

$$- \sum_{\substack{0\leq l,m\leq Tp^{-1}+1 \\ l\neq m}} \mathbf{P}\left(\max_{\Delta_l} \zeta_L^{(N)}(k) > u, \max_{\Delta_m} \zeta_L^{(N)}(k) > u\right).$$

Let $p_{l,m}$ denote the summands in the last sum. This sum, due to stationarity, does not exceed

$$2(Tp^{-1}+1)\sum_{n=1}^{Tp^{-1}+1} p_{0,n}.$$

Estimating the probabilities $p_{0,n}$ from above we will show that the double sum is negligible, and thus the upper and lower estimates in the Bonferroni inequality are asymptotically equivalent. The estimates of $p_{0,n}$ are obtained in the same manner as in Piterbarg (1991). The proof will be divided into four parts.

Case 1.1: $1 \leq n < n_0$. The value of n_0 will be chosen later. We have

$$p_{0,n} \leq \mathbf{P}\left(\max_{\substack{0<k\leq pN(n+1)/2 \\ (1-q)N<L\leq N}} \zeta_L^{(N)}(k) > u, \max_{\substack{pN(n+1)/2<k\leq pN(n+1) \\ (1-q)N<L\leq N}} \zeta_L^{(N)}(k) > u\right)$$

$$= 2\mathbf{P}\left(\max_{\substack{0<k\leq pN(n+1)/2 \\ (1-q)N<L\leq N}} \zeta_L^{(N)}(k) > u\right) - \mathbf{P}\left(\max_{\substack{0<k\leq pN(n+1) \\ (1-q)N<L\leq N}} \zeta_L^{(N)}(k) > u\right).$$

Applying Lemma 2.1 we obtain that

$$p_{0,n} \leq \frac{1}{\sqrt{2\pi}u}(2H^B_{A(n+1)/2} - H^B_{A(n+1)})e^{-\frac{u^2}{2}}(1+g_n(u,N)), \tag{4}$$

where $g_n(u,N) \to 0$.

Case 1.2: $n_0 \leq n \leq \varepsilon p^{-1} - 1$. The value of ε will be chosen later. First, introduce random variables

$$\eta = \tfrac{1}{\sqrt{N}} \sum_{i=(n+1)pN+1}^{(1-q)N} \xi_i, \quad \zeta_1 = \tfrac{1}{\sqrt{N}} \sum_{i=pN+1}^{npN} \xi_i, \quad \zeta_2 = \tfrac{1}{\sqrt{N}} \sum_{i=pN+N+1}^{npN+(1-q)N} \xi_i.$$

Then, postponing the explanation of the last equality,

$$p_{0,n} = \mathbf{P}\Bigg(\eta + \zeta_1 + \max_{\Delta_0} \tfrac{1}{\sqrt{N}} \left(\sum_{i=k}^{pN} + \sum_{i=npN+1}^{(n+1)pN} + \sum_{i=(1-p)N+1}^{k+L-1} \right) \xi_i > u,$$

$$\eta + \zeta_2 + \max_{\Delta_n} \tfrac{1}{\sqrt{N}} \left(\sum_{i=k}^{(n+1)pN} + \sum_{i=(1-q)N+1}^{pN+N} + \sum_{i=npN+(1-q)N+1}^{k+L-1} \right) \xi_i > u \Bigg)$$

$$= \mathbf{P}\left(\eta + \zeta_1 + \max_{\Delta_0} \zeta_L^{'}(k) > u, \quad \eta + \zeta_2 + \max_{\Delta_0} \zeta_L^{''}(k) > u \right),$$

where

$$\zeta_L^{'}(k) = \frac{1}{\sqrt{N}} \left(\sum_{i=k}^{k+L-(1-q-2p)N-1} \xi_i^{'} \right) \quad \text{and} \quad \zeta_L^{''}(k) = \frac{1}{\sqrt{N}} \left(\sum_{i=k}^{k+L-(1-2q-2p)N-1} \xi_i^{''} \right). \tag{5}$$

The main idea of this representation is that we consider $\zeta_L^{(N)}(k)$ for all possible pairs $(k, L) \in \Delta_0$ and "extract" the common summand $\eta + \zeta_1$. Analogously, for each $(k, L) \in \Delta_n$ we "extract" the summand $\eta + \zeta_2$. These summands are always present in $\zeta_L^{(N)}(k)$ when k, L are within the corresponding sets. It is easy to check that for $\varepsilon < 1/2$ and u large, the restriction on n ensures that the variables $\eta,\ \zeta_1,\ \zeta_2$ are independent. By construction they are also independent of the variables that remain inside the maximum signs. The latter are renumbered and called ξ_i' and ξ_i'' in such a way that (5) holds. Although ξ_i' and ξ_j'' may denote the same r.v. ξ_r, in our case the dependence between $\zeta_L'(k)$ and $\zeta_L''(k)$ does not matter. What is essential is that $\eta,\ \zeta_1,\ \zeta_2$, are independent of $\zeta_L'(k)$ and of $\zeta_L''(k)$. We now omit the arguments for $\zeta_L'(k)$ and $\zeta_L''(k)$, and the set over which the maximum is taken.

From (5) it follows that

$$p_{0,n} \le \mathbf{P}\left(\eta + \tfrac{\zeta_1+\zeta_2}{2} + \tfrac{\max\zeta' + \max\zeta''}{2} > u \right) = \tfrac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \mathbf{P}\left(\tfrac{\zeta_1 + \zeta_2}{2} + \tfrac{\max\zeta' + \max\zeta''}{2} > u - v \right) e^{-\frac{v^2}{2\sigma^2}}\, dv,$$

where σ^2 now is equal to $\frac{[(1-q)N]-[(n+1)pN]}{N}$. Changing variables $v = u - \sqrt{p}s$ we get

$$\begin{aligned} p_{0,n} &\le \tfrac{\sqrt{A}}{\sqrt{2\pi\sigma^2}u} e^{-\frac{u^2}{2\sigma^2}} \int_{-\infty}^{\infty} \mathbf{P}\left(\tfrac{\zeta_1+\zeta_2}{2\sqrt{p}} + \tfrac{\max\zeta' + \max\zeta''}{2\sqrt{p}} > s \right) e^{\frac{\sqrt{A}s}{\sigma^2}}\, ds \\ &= \tfrac{\sigma}{\sqrt{2\pi}u} e^{-\frac{u^2}{2\sigma^2}} \mathbf{E} e^{\frac{\sqrt{A}}{\sigma^2}\left(\frac{\zeta_1+\zeta_2}{2\sqrt{p}} + \frac{\max\zeta' + \max\zeta''}{2\sqrt{p}} \right)} \\ &= \tfrac{\sigma}{\sqrt{2\pi}u} e^{-\frac{u^2}{2\sigma^2}} \mathbf{E} e^{\frac{\sqrt{A}}{\sigma^2}\left(\frac{\zeta_1+\zeta_2}{2\sqrt{p}} \right)} \mathbf{E} e^{\frac{\sqrt{A}}{\sigma^2}\left(\frac{\max\zeta' + \max\zeta''}{2\sqrt{p}} \right)}. \end{aligned} \tag{6}$$

We now estimate the three factors that form the bound for $p_{0,n}$. Since $\sigma^2 = 1 - q - (n+1)p + o(u^{-2})$, for sufficiently large u the factor in front of the expectation is bounded by

$$\frac{\sigma}{\sqrt{2\pi}u} e^{-\frac{u^2}{2\sigma^2}} \le \frac{2}{\sqrt{2\pi}u} e^{-\frac{u^2}{2}} e^{-\frac{A(n+1)+B}{2}}.$$

Next, since random variable $\frac{\zeta_1+\zeta_2}{2\sqrt{p}}$ is normally distributed, has zero mean and its variance does not exceed $(n-1)/2$, we have that

$$\mathbf{E}e^{\frac{\sqrt{A}}{\sigma^2}\left(\frac{\zeta_1+\ \zeta_2}{2\sqrt{p}}\right)} \le e^{\frac{A(n-1)}{4\sigma^4}}.$$

In order to estimate the remaining expectation we will require an estimate of the probability

$$\mathbf{P}\left(\frac{\max\zeta'+\max\zeta''}{2\sqrt{p}} > s\right), \quad s>0.$$

According to notation in (5) and denoting $S_k^{''} = \sum_{i=1}^{k}\xi_i^{''}$ we see that the latter equals

$$\begin{aligned}
&\mathbf{P}\left(\max_{\Delta_0}\left(S^{'}_{k+L-(1-q-2p)N-1}-S^{'}_{k-1}\right)+\max_{\Delta_0}\left(S^{''}_{k+L-(1-2q-2p)N-1}-S^{''}_{k-1}\right)>2\sqrt{pN}s\right)\\
&\qquad\le \mathbf{P}\left(\max_{\Delta_0}S^{'}_{k+L+(q+2p)N-N-1}+\max_{0<k\le pN}-S^{'}_{k-1}\right.\\
&\qquad\qquad\left.+\max_{\Delta_0}S^{''}_{k+L+(2q+2p)N-N-1}+\max_{0<k\le pN}-S^{''}_{k-1}>2\sqrt{pN}s\right)\\
&\qquad\le 4\mathbf{P}\left(\max_{0<k\le(2q+3p)N}S^{'}_{k}>\frac{\sqrt{pN}}{2}s\right)\le 4e^{-\frac{1}{8}\left(\frac{A}{3A+2B}s^2\right)}<4e^{-\frac{s^2}{24}},
\end{aligned} \tag{7}$$

where we applied Lemma 1 Piterbarg (1991) in the second to the last step. Thus, for any positive t we obtain the following estimate

$$\mathbf{E}e^{t\left(\frac{\max\zeta'+\ \max\zeta''}{2\sqrt{p}}\right)} = \int_{-\infty}^{\infty} te^{ts}\mathbf{P}\left(\frac{\max\zeta'+\max\zeta''}{2\sqrt{p}}>s\right)ds \le 1+4t\int_0^\infty e^{ts-\frac{s^2}{24}}\,ds \le 1+4\sqrt{24\pi}te^{6t^2}. \tag{8}$$

Then we put $t=\frac{\sqrt{A}}{\sigma^2}$ and when A is large, the estimate (8) gives

$$\mathbf{E}e^{\frac{\sqrt{A}}{\sigma^2}\left(\frac{\max\zeta'+\ \max\zeta''}{2\sqrt{p}}\right)} < \frac{8\sqrt{24\pi}}{\sigma^2}\sqrt{A}e^{\frac{6A}{\sigma^4}}.$$

We are now ready to estimate $p_{0,n}$. Gathering the estimates of the factors in (6) we get

$$p_{0,n} \le \frac{\frac{16\sqrt{24\pi}}{\sigma^2}\sqrt{A}}{\sqrt{2\pi}u}e^{-\frac{u^2}{2}}e^{-An\left(\frac{1}{2}-\frac{1}{4\sigma^4}\right)+A\left(\frac{23}{4\sigma^4}-\frac{1}{2}\right)-\frac{B}{2}}.$$

Owing to the restriction $n_0\le n\le \varepsilon p^{-1}-1$ we have $\sigma^2 = 1-q-(n+1)p+o(u^{-2}) > 1-2\varepsilon$, and choosing ε such that $4(1-2\varepsilon)^2=3$ we conclude that

$$p_{0,n} \le \frac{C_1\sqrt{A}}{\sqrt{2\pi}u}e^{-\frac{u^2}{2}}e^{-A\frac{n-43}{6}-\frac{B}{2}}, \tag{9}$$

where C_1 is some positive constant.

Case 1.3: $\varepsilon p^{-1}\le n\le p^{-1}+1$. In much the same way the representation (5) gives

$$p_{0,n} \le \mathbf{P}\left(2\eta+\zeta_1+\zeta_2+\max_{\Delta_0}\zeta_L'(k)+\max_{\Delta_0}\zeta_L''(k)>2u\right).$$

However, when $n\ge\varepsilon p^{-1}$, it may turn out that the sum in the expression for η is empty. In this case we set $\eta=0$. We should also change the upper limit of summation in the definition of ζ_1 to $\min\{npN,(1-p)N\}$, and the lower limit of summation for ζ_2 to $\max\{2pN+(1-p)N+1,(n+1)pN+1\}$. Therefore, ζ' and ζ'', may consist of a smaller number of summands.

For any positive t, multiplying both parts of the inequality under the probability sign by t/2 and applying Chebyshev's inequality to the exponents, we obtain that

$$\begin{aligned} p_{0,n} &\le e^{-tu}\mathbf{E}e^{t\left(\eta+\frac{\zeta_1+\zeta_2}{2}+\frac{\max\zeta'+\max\zeta''}{2}\right)} \\ &= e^{-tu}\mathbf{E}e^{t\left(\eta+\frac{\zeta_1+\zeta_2}{2}\right)}\mathbf{E}e^{t\left(\frac{\max\zeta'+\max\zeta''}{2}\right)}. \end{aligned} \tag{10}$$

Although ζ' and ζ'' may contain smaller number of summands, it can be seen that this does not change the proof of (7) sufficiently. Thus the estimate (8) remains valid and

$$\mathbf{E}e^{t\left(\frac{\max\zeta'+\max\zeta'}{2}\right)} < 1+4\sqrt{24\pi}t\sqrt{p}e^{6t^2p}. \tag{11}$$

Next, according to the remark about limits of summation in ζ_1 and ζ_2, the variance of $\frac{\zeta_1+\zeta_2}{2}$ does not exceed $\frac{(n-1)p}{2}$. The variance of η does not exceed $\max\{0,1-(n-1)p\}$. Applying Laplace transformation to the sum $\eta+\frac{\zeta_1+\zeta_2}{2}$, and since restrictions on n provide $(\varepsilon-p)/2\le(n-1)p/2\le 1/2$,

$$\mathbf{E}e^{t\left(\eta+\frac{\zeta_1+\zeta_2}{2}\right)} \le e^{\frac{t^2\max\left\{\frac{(n-1)p}{2},1-\frac{(n-1)p}{2}\right\}}{2}} < e^{\frac{t^2(1-\varepsilon/4)}{2}}. \tag{12}$$

So, gathering (12), (11) and (10),

$$p_{0,n}\le(1+4\sqrt{24\pi}t\sqrt{p}e^{6t^2p})e^{\frac{t^2(1-\varepsilon/4)}{2}}e^{-tu}.$$

Setting $t=\frac{u}{1-\varepsilon/4}$, we obtain the desired estimate

$$p_{0,n}\le C_2\sqrt{A}e^{6A}e^{-\frac{u^2}{2(1-\frac{\varepsilon}{4})}}. \tag{13}$$

Case 1.4: $n>p^{-1}+1$. In this case the two events inside the probability $p_{0,n}$ are independent and Lemma 2.1 gives

$$p_{0,n}\le 2(H_A^B)^2\psi(u)^2. \tag{14}$$

The bounds obtained in cases 1.1-1.4 allow us to estimate $p_{0,n}$ for any value of n. Let $p_0(u)=\frac{1}{\sqrt{2\pi}u}e^{-\frac{1}{2}u^2}$. Estimates (4), (9), (13), (14) imply that

$$\begin{aligned} 2(Tp^{-1}+1)\sum_{n=1}^{Tp^{-1}+1}p_{0,n} &\le 2(Tp^{-1}+1) \\ &\times\left\{\left(\sum_{n=1}^{n_0-1}\left(2H^B_{A(n+1)/2}-H^B_{A(n+1)}\right)(1+g_n(u,N))+\sum_{n=n_0}^{\infty}C_1\sqrt{A}e^{-A\frac{n-43}{6}-\frac{B}{2}}\right)p_0(u)\right. \\ &\left.+p^{-1}C_2\sqrt{A}e^{6A}e^{-\frac{u^2}{2(1-\frac{\varepsilon}{4})}}+Tp^{-1}2(H_A^B)^2\psi(u)^2\right\}. \end{aligned}$$

Recall that $p^{-1}=u^2/A$. If $Tu^2\to\infty$ and $Tu^2\psi(u)\to 0$, then using the estimate above and the Bonferroni inequality on page 5 we conclude that

$$\overline{\lim_{u,N}}\,\mathbf{P}\left(\max_{\substack{0<k\le TN\\(1-q)N<L\le N}}\zeta_L^{(N)}(k)>u\right)\Big/ Tu^2p_0(u)\le A^{-1}H_A^B$$

and (15)

$$\varliminf_{u,N} \mathbf{P}\left(\max_{\substack{0<k\le TN \\ (1-q)N<L\le N}} \zeta_L^{(N)}(k) > u \right) \Big/ Tu^2 p_0(u) \ge A^{-1} H_A^B - 2A^{-1} \sum_{n=1}^{n_0-1} \left(2H^B_{A(n+1)/2} - H^B_{A(n+1)} \right) - 2\frac{C_1 e^{-\frac{B}{2}}}{\sqrt{A}} \sum_{n=n_0}^{\infty} e^{-A\frac{n-43}{6}}.$$

It was proved in Zholud (2008) that the limit

$$H^B = \lim_{A\to\infty} A^{-1} H_A^B, \quad H^B > 0$$

exists. Thus $A^{-1}\left(2H^B_{A(n+1)/2} - H^B_{A(n+1)}\right) \to 0$, when $A \to \infty$. Choosing n_0 greater than 43 and letting A in (15) tend to infinity we obtain the asymptotic behavior of the probability of high level excursions for maximum of $\zeta_L^{(N)}(k)$ over the “upper” layer,

$$\mathbf{P}\left(\max_{\substack{0<k\le TN \\ (1-q)N<L\le N}} \zeta_L^{(N)}(k) > u \right) = H^B T u^2 p_0(u)(1+o(1)). \tag{16}$$

The second part of the proof is to show that the asymptotic behavior of the probability (1) is determined by the behavior of $\zeta_L^{(N)}(k)$ over the upper layer, which corresponds to the area of maximal variance of the field. Thus we need to estimate the probability of the high level excursion of the maximum of the random walk over the complementary set. Applying stationarity of $\zeta_L^{(N)}(k)$ with respect to k we obtain the following estimate

$$\mathbf{P}\left(\max_{\substack{0<k\le TN \\ 0<L\le(1-q)N}} \zeta_L^{(N)}(k) > u \right) \le (Tp^{-1}+1) \sum_{n=1}^{p^{-1}-1} \times \mathbf{P}\left(\max_{\substack{0<k\le pN \\ (1-(n+1)q)N<L\le(1-nq)N}} \zeta_L^{(N)}(k) > u \right). \tag{17}$$

Let p_n denote the probability under the sum sign. Bounds for p_n will be obtained in two steps.

Case 2.1: $n < \frac{13}{16}p^{-1} - 1$. The restriction on n ensures that the sum extracted from $\zeta_L^{(N)}(k)$ in the equality below is not empty

$$\max_{\substack{0<k\le pN \\ (1-(n+1)q)N<L\le(1-nq)N}} \zeta_L^{(N)}(k) = \frac{1}{\sqrt{N}} \sum_{i=[pN]+1}^{[(1-(n+1)q)N]} \xi_i + \max_{\substack{0<k\le pN \\ (1-(n+1)q)N<L\le(1-nq)N}} \frac{1}{\sqrt{N}} \left(\sum_{i=k}^{[pN]} \xi_i + \sum_{i=[(1-(n+1)q)N]+1}^{k+L-1} \xi_i \right).$$

Repeating the proof of Lemma 2.1 we obtain the following analog of the equality (3),

$$p_n = \frac{\sqrt{A}}{\sqrt{2\pi\sigma'^2}u} e^{-\frac{u^2}{2\sigma'^2}} \int_{-\infty}^{\infty} e^{-\frac{Aw^2/u^2}{2\sigma'^2}} e^{\frac{\sqrt{A}w}{\sigma'^2}} \mathbf{P}\left(M(k,L) > w\right) dw, \tag{18}$$

where σ'^2 is equal to $\frac{[(1-(n+1)q)N]-[pN]}{N}$.

The expression (3) for the probability in Lemma 2.1 differs from (18) only in the variance σ'^2 of the extracted summand. Recall that σ^2 in Lemma 2.1 is equal to $\frac{[(1-q)N]-[pN]}{N}$. It is straightforward to show that

$$\frac{\sigma^2}{\sigma'^2} = 1 + \frac{nq}{1-(n+1)q-p} + o(u^{-2}) = 1 + z.$$

With this notation the right-hand side of (18) takes form

$$\frac{\sqrt{A}e^{-\frac{u^2}{2\sigma^2}(1+z)}}{\sqrt{2\pi\sigma'^2}u} \int_{-\infty}^{\infty} e^{-\frac{Aw^2/u^2}{2\sigma^2}(1+z)+\frac{\sqrt{A}w}{\sigma^2}z} e^{\frac{\sqrt{A}w}{\sigma^2}} \mathbf{P}\left(M(k,L) > w\right) dw.$$

The first exponent under the integral sign is a parabola with respect to w and attains its maximum at the point $w = \frac{z}{z+1}\frac{u^2}{\sqrt{A}}$. Straightforward calculation then show that

$$p_n \le \frac{\sqrt{A}}{\sqrt{2\pi\sigma'^2}u} e^{-\frac{u^2}{2\sigma^2}K} \int_{-\infty}^{\infty} e^{\frac{\sqrt{A}w}{\sigma^2}} \mathbf{P}\left(M(k,L) > w\right) dw,$$

where

$$K = 1 + \frac{z}{1+z} = 1 + \frac{nq}{1-q-p} \ge 1 + nq.$$

Finally, owing to Lemma 2.1 there exists a constant C such that

$$p_n \le \frac{\sigma}{\sigma'} e^{-\frac{nB}{2}} H_A^B \frac{1}{\sqrt{2\pi}u} e^{-\frac{u^2}{2}}(1+o(1)) \le Ce^{-\frac{nB}{2}} H_A^B p_0(u),$$

where $o(1) \to 0$ uniformly in n when $u, N \to \infty$.

Case 2.2: $np \ge \frac{13}{16}$. Now σ'^2 can be arbitrary small and using Lemma 1 of Piterbarg (1991) we get

$$p_n \le \mathbf{P}\left(\max_{\substack{0<k\le pN \\ 0<L\le \frac{3}{16}N}} \zeta_L^{(N)}(k) > u\right) \le 2\mathbf{P}\left(\max_{0<k\le \frac{3}{16}N+pN} S_k > \tfrac{1}{2}u\sqrt{N}\right) \le 2e^{-\frac{u^2}{4(\frac{3}{16}+p)}} \le 2e^{-u^2}.$$

Thus, combining the estimates for p_n obtained in cases 2.1 and 2.2 with (17) and (16) we have

$$\overline{\lim_{u,N}} \mathbf{P}\left(\max_{\substack{0<k\le TN \\ 0<L\le N}} \zeta_L^{(N)}(k) > u\right) \Big/ (Tu^2 p_0(u)) \le H^B + \frac{H_A^B C}{A} \sum_{n=1}^{\infty} e^{-\frac{nB}{2}},$$

$$\underline{\lim_{u,N}} \mathbf{P}\left(\max_{\substack{0<k\le TN \\ 0<L\le N}} \zeta_L^{(N)}(k) > u\right) \Big/ (Tu^2 p_0(u)) \ge H^B.$$

It was proved in Zholud (2008) that the limit $H = \lim_{B\to\infty} H^B$ exists and is positive. Letting first A, and then B tend to infinity, we conclude that the upper and lower limits coincide and equal H. This finishes the proof of Theorem 1.3.

3 Very large deviations of the Shepp statistic

Here we prove Theorem 1.4. The asymptotic behavior of the probability (1) under assumption that $u/\sqrt{N} \to \infty$ is considered. First, we find the asymptotic behavior of the probability

$$\mathbf{P}\left(\max_{0<k\le TN} \zeta_N^{(N)}(k) > u\right). \tag{19}$$

As in the previous section, we then show that the maximum of the field $\zeta_L^{(N)}(k)$ over the complementary set $\{(k,L) : 0 < k \leq TN,\ 0 < L \leq N-1\}$ gives neglible contribution to the probability (1).

Now a key lemma that plays an essential role in establishing the asymptotic formula for (19).

Lemma 3.1. *Let (ξ_1, ξ_2) be a Gaussian random vector such that ξ_1 and ξ_2 are standard normal variables with correlation coefficient $\alpha < 1$. Then,*

$$\mathbf{P}\left(\xi_1 > u, \xi_2 > u\right) < \frac{1}{\sqrt{2\pi}u} e^{-\frac{1}{2}u^2} \frac{1}{\sqrt{2\pi}u}(1+\alpha)\frac{\sqrt{1+\alpha}}{\sqrt{1-\alpha}} e^{-\frac{1}{2}u^2\frac{1-\alpha}{1+\alpha}}.$$

Proof: The variable ξ_2 can be expressed as the sum of two independent variables $\alpha\xi_1$ and ζ, where $\zeta \sim N(0, 1-\alpha^2)$. By $\varphi_\zeta(\cdot)$ we will refer to the density function of ζ. Denoting the probability in the statement of the lemma by $I(u)$ we have

$$\begin{aligned} I(u) &= \mathbf{P}\left(\xi_1 > u, \alpha\xi_1 + \zeta > u\right) = \tfrac{1}{\sqrt{2\pi}} \int_u^\infty e^{-\frac{v^2}{2}} \mathbf{P}(\zeta > u - \alpha v)\, dv = -\tfrac{1}{\sqrt{2\pi}} \int_u^\infty \tfrac{\mathbf{P}(\zeta > u-\alpha v)}{v}\, de^{-\frac{v^2}{2}} \\ &= -\tfrac{\mathbf{P}(\zeta>u-\alpha v)}{\sqrt{2\pi}v} e^{-\frac{v^2}{2}}\Big|_u^\infty + \tfrac{1}{\sqrt{2\pi}} \int_u^\infty e^{-\frac{v^2}{2}}\, d\tfrac{\mathbf{P}(\zeta>u-\alpha v)}{v} \\ &= \tfrac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}u} \mathbf{P}\left(\zeta > u(1-\alpha)\right) + \int_u^\infty \tfrac{e^{-\frac{v^2}{2}}}{\sqrt{2\pi}} \Big(\alpha \tfrac{\varphi_\zeta(u-\alpha v)}{v} - \tfrac{\mathbf{P}(\zeta>u-\alpha v)}{v^2}\Big) dv. \end{aligned}$$

Write $K(u)$ for the first summand in the last expression. The second summand is less than

$$\frac{\alpha}{\sqrt{2\pi}u} \int_u^\infty e^{-\frac{v^2}{2}} \varphi_\zeta(u - \alpha v)\, dv$$

and thus $I(u)$ is bounded by

$$\begin{aligned} K(u) + \tfrac{\alpha}{\sqrt{2\pi}u} \int_u^\infty \tfrac{1}{\sqrt{2\pi(1-\alpha^2)}} e^{-\frac{1}{2}\left(v^2 + \frac{(u-\alpha v)^2}{1-\alpha^2}\right)} dv &= K(u) + \alpha \tfrac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}u} \int_u^\infty \tfrac{1}{\sqrt{2\pi(1-\alpha^2)}} e^{-\frac{1}{2}\frac{(v-\alpha u)^2}{1-\alpha^2}} dv \\ = K(u) + \alpha K(u) = \tfrac{1}{\sqrt{2\pi}u} e^{-\frac{1}{2}u^2}(1+\alpha)\mathbf{P}\left(\tfrac{\zeta}{\sqrt{1-\alpha^2}} > u\tfrac{\sqrt{1-\alpha}}{\sqrt{1+\alpha}}\right). \end{aligned}$$

The lemma now follows from the standard upper bound of the standard normal distribution tail. □

Next, we estimate (19) using the Bonferroni inequality

$$\begin{aligned} [TN]\mathbf{P}\left(\zeta_N^{(N)}(1) > u\right) &\geq \mathbf{P}\left(\max_{0<k\leq TN} \zeta_N^{(N)}(k) > u\right) \\ &\geq [TN]\mathbf{P}\left(\zeta_N^{(N)}(1) > u\right) - \sum_{\substack{1\leq l,m\leq TN \\ l\neq m}} \mathbf{P}\left(\zeta_N^{(N)}(l) > u, \zeta_N^{(N)}(m) > u\right). \end{aligned}$$

By stationarity, and applying Lemma 3.1 with

$$\alpha = \alpha_n = \mathbf{E}\zeta_N^{(N)}(1)\zeta_N^{(N)}(n) = \max\{0, \frac{N-(n-1)}{N}\},$$

we get that the double sum is bounded by

$$\begin{aligned} 2TN \sum_{n=2}^{TN} \mathbf{P}\left(\zeta_N^{(N)}(1) > u, \zeta_N^{(N)}(n) > u\right) &< 2TN \sum_{n=N+1}^{TN} \mathbf{P}\left(\zeta_N^{(N)}(1) > u\right)^2 \\ &+ 2TN \sum_{n=2}^{N} \tfrac{1}{\sqrt{2\pi}u} e^{-\frac{1}{2}u^2} \tfrac{1}{\sqrt{2\pi}u}(1+\alpha_n)\tfrac{\sqrt{1+\alpha_n}}{\sqrt{1-\alpha_n}} e^{-\frac{1}{2}u^2\frac{1-\alpha_n}{1+\alpha_n}}. \end{aligned}$$

As before let $p_0(u)$ denote $\frac{1}{\sqrt{2\pi}u}e^{-\frac{1}{2}u^2}$, the asymptotic bound for the standard normal distribution tail. The first summand is then less than

$$2(TN)^2\mathbf{P}\left(\zeta_N^{(N)}(1) > u\right)^2 = 2(TN)^2 p_0(u)^2(1+o(1))$$

and the second is estimated from above by

$$2TNp_0(u)\frac{2\sqrt{2N}}{\sqrt{2\pi}u}\sum_{n=2}^{N}\left(e^{-\frac{u^2/N}{4}}\right)^{n-1} = o(TNp_0(u)),$$

where we took into account that $u/\sqrt{N} \to \infty$.

Replacing the double sum by its upper estimate and dividing both sides of the Bonferroni inequality by $[TN]p_0(u)$, and assuming $TN \geq 1$, we get that

$$1 + o(1) \geq \mathbf{P}\bigg(\max_{0<k\leq TN} \zeta_N^{(N)}(k) > u\bigg)\bigg/ [TN]p_0(u) \geq 1 - 4TNp_0(u)(1+o(1)) + o(1).$$

Finally, for $TNp_0(u) \to 0$ we obtain the following asymptotic formula for the probability (19),

$$\mathbf{P}\left(\max_{0<k\leq TN} \zeta_N^{(N)}(k) > u\right) = [TN]p_0(u)(1+o(1)). \tag{20}$$

The remaining step is to note that the probability for the maximum over the complementary set is neglible. Since

$$\begin{aligned}\mathbf{P}\left(\max_{\substack{0<k\leq TN\\0<L\leq N-1}} \zeta_L^{(N)}(k) > u\right) &\leq TN\sum_{L=1}^{N-1}\mathbf{P}\left(\zeta_L^{(N)}(1) > u\right)\\ &\leq TN\sum_{L=1}^{N-1} p_0\left(u\sqrt{\frac{N}{L}}\right) = TNp_0(u)\sum_{L=1}^{N-1} e^{-\frac{u^2(N-L)}{2L}}\\ &\leq TNp_0(u)\sum_{L=1}^{N-1}\left(e^{-\frac{u^2/N}{2}}\right)^{N-L} = o(TNp_0(u)),\end{aligned}$$

the latter estimate and (20) conclude the proof of Theorem 1.4.

4 Limit theorems for $M_T^{(N)}$

In this section we consider the case when T, N go to infinity. It can be shown that for appropriate normalization constants a_T and b_T the limit distribution of $\left(M_T^{(N)} - a_T\right)\big/ b_T$ is Gumbel. Theorem 4.1 exhibits the normalizing constants for three different limit relations between T and N.

Theorem 4.1. *Assume that one of the following relations hold:*

$$1)\ \frac{2\ln T}{N} \to 0. \quad 2)\ \frac{2\ln T}{N} \to \theta^2 > 0. \quad 3)\ \frac{2\ln T}{N} \to \infty.$$

Then, for any fixed x,

$$\mathbf{P}\left(\max_{\substack{0<k\leq TN\\0<L\leq N}} a_T(\zeta_L^{(N)}(k) - b_T) \leq x\right) = e^{-e^{-x}} + o(1),$$

where

$$a_T = \sqrt{2\ln T}, \qquad b_T = \sqrt{2\ln T} + \frac{F(T,N) + \frac{1}{2}(\ln\ln T - \ln\pi)}{\sqrt{2\ln T}}$$

and the function $F(T,N)$ *is given by*

$$1)\ F(T,N) = \ln H \quad 2)\ F(T,N) = \ln\frac{J_\theta}{\theta} \quad 3)\ F(T,N) = -\ln\frac{2\ln T}{N}.$$

The proof follows from Lemma 3.1 of Zholud (2008) closely, and is hence omitted.

The limit distribution for the case $\frac{2\ln T}{N} = \theta^2,\ 0 < \theta < \infty$ was obtained by A.M. Kozlov in Kozlov (2004) and was reformulated in Theorem 4.1 for comparison purpose.

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